

# Multiplicity estimate for solutions of extended Ramanujan's system.

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## Abstract

We establish a new *multiplicity lemma* for solutions of a differential system extending Ramanujan's classical differential relations. This result can be useful in the study of arithmetic properties of values of Riemann zeta function at odd positive integers (Nesterenko, 2011).

## 1 Introduction

In what follows we denote by  $\sigma_k(n)$ ,  $k \in \mathbb{Z}$ ,  $n \in \mathbb{N}$  the sum of  $k$ th powers of divisors of  $n$ :

$$\sigma_k(n) := \sum_{d|n} d^k.$$

In this paper we consider the following sets of functions. First of all, the Eisenstein series

$$E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) z^n, \quad k \in \mathbb{N}, \quad (1)$$

where  $B_{2k}$  are Bernoulli numbers. Also we consider

$$g_{u,v}(z) := \sum_{n=1}^{\infty} n^u \sigma_{-v}(n) z^n, \quad 0 \leq u < v, \quad , u, v \in \mathbb{N}.$$

It is well-known that functions  $E_2$ ,  $E_4$  and  $E_6$  are algebraically independent over  $\mathbb{C}(z)$  and all the other functions  $E_{2k}$ ,  $k \geq 4$  can be expressed in terms of  $E_4$  and  $E_6$  (see for instance [6]). More precisely, for all  $k \geq 4$  there exists a polynomial  $A_k \in \mathbb{C}[X, Y]$  such that

$$E_{2k}(z) = A_k(E_4(z), E_6(z)).$$

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These polynomials  $A_k(X, Y)$ ,  $k \geq 4$  contain only monomials  $M$  of bi-degrees  $(\deg_X M, \deg_Y M)$  satisfying  $2 \deg_X M + 3 \deg_Y M = k$ .

In 2010 P.Kozlov proved (see [2], page 2) that for any fixed  $m \in \mathbb{N}$  all the functions

$$E_2(z), E_4(z), E_6(z), g_{u,v}(z), \quad 0 \leq u < v \leq m \quad (2)$$

are algebraically independent over  $\mathbb{C}(z)$ .

The functions (2) satisfy the following system of differential equations [2]. Denote  $\delta := z \frac{d}{dz}$ . Then

$$\delta E_2 = \frac{1}{12} (E_2^2 - E_4), \delta E_4 = \frac{1}{3} (E_2 E_4 - E_6), \delta E_6 = \frac{1}{2} (E_2 E_6 - E_4^2) \quad (3)$$

and for any odd  $v \geq 3$

$$\begin{aligned} \delta g_{u,v}(z) &= g_{u+1,v}(z), \quad 0 \leq u < v-1, \\ \delta g_{v-1,v}(z) &= B_{2v+2} \frac{A_{v+1}(E_4(z), E_6(z)) - 1}{2v+2}. \end{aligned} \quad (4)$$

In the case  $v = 1$  one has

$$\delta g_{0,1}(z) = \frac{1}{24} (1 - E_2(z)). \quad (5)$$

Yu.Nesterenko [2] showed that values of functions  $g_{u,v}(z)$  are closely related to the values of the Riemann zeta function  $\zeta$  at odd positive integers. In particular,  $\zeta(4n+3) \in \mathbb{Q}(E_2(i), g_{0,4n+3}(i))$  [2]. Whereas the system (3), (4), (5) for functions  $E_2, E_4, E_6, (g_{u,v})_{0 \leq u < v \leq m}$ ,  $m \in \mathbb{N}$ , is quite a simple extension of the system (3), and in the case of the system (3) Nesterenko [1] established an optimal algebraic independence result for its solutions [1], one may hope that this approach will lead to some results concerning algebraic independence of values of  $\zeta$  at positive integral odd points. On this way, an important stage is a *multiplicity lemma* for the functions in question.

In this paper we adopt the method from [1] and [3][Chapter 10] to establish (for any fixed odd  $m \geq 3$ ) a multiplicity lemma for the whole set of functions  $E_2, E_4, E_6, (g_{u,v}), 0 \leq u < v \leq m$ , see Theorem 2.1 below.

## 2 Multiplicity Lemma

Let  $m \in \mathbb{N}$  be a fixed positive odd integer. We introduce the following notation:

$$R := \mathbb{C}[X_0, X_1, X_2, X_3, Y_{0,1}, Y_{0,3}, Y_{1,3}, Y_{2,3}, \dots, Y_{m-1,m}].$$

**Theorem 2.1** *Let  $m \geq 1$  be an odd integer. For all non-zero  $P \in R$  there exists a constant  $C$  depending on  $m$  only such that*

$$\begin{aligned} \text{ord}_{z=0} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), g_{0,3}(z), \dots, g_{0,m}(z), \dots, g_{m-1,m}(z)) \\ \leq C (\deg_{X_0} P + 1) (\deg_{Eg} P + 1)^{\left(\frac{m-1}{2}\right)^2 + 3}, \end{aligned} \quad (6)$$

where  $\deg_{Eg} P$  denotes the total degree of  $P$  in the variables  $X_1, X_2, X_3, Y_{0,1}, \dots, Y_{m-1,m}$ , i.e. all the variables appearing in  $R$  but  $X_0$ .

**Remark 2.2** *The exponent  $\left(\frac{m-1}{2}\right)^2 + 3$  in the r.h.s. of (6) equals the number of functions different than  $z$  in the l.h.s. of (6) and also the transcendence degree of  $R$  over  $\mathbb{C}(z)$ . Hence Theorem 2.1 provides multiplicity estimate with the optimal exponent.*

In the sequel we denote

$$D_0 := z \frac{d}{dz} + \frac{1}{12} (X_1^2 - X_2) \frac{d}{dX_1} + \frac{1}{3} (X_1 X_2 - X_3) \frac{d}{dX_2} + \frac{1}{2} (X_1 X_3 - X_2^2) \frac{d}{dX_3},$$

$$D_1 := \frac{1}{24} (1 - X_2) \frac{d}{dY_{0,1}},$$

$$D_v := \sum_{k=0}^{v-2} Y_{k+1,v} \frac{d}{dY_{k,v}} + B_{v+1} \frac{A_{v+1}(X_2, X_3) - 1}{2v + 2} \frac{d}{dY_{v-1,v}}, \quad v = 3, 5, \dots, m$$

and

$$D := D_0 + \sum_{k=0}^{(m-1)/2} D_{2k+1}. \quad (7)$$

The differential operator  $D$  satisfies

$$\begin{aligned} DP(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) \\ = z \frac{d}{dz} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)). \end{aligned} \quad (8)$$

We deduce Theorem 2.1 using Nesterenko's conditional Multiplicity Lemma (Theorem 1.1, Chapter 10 [3]). This result deals with differential system of the following type:

$$f'_i(z) = \frac{A_i(z, \underline{f})}{A_0(z, \underline{f})}, \quad i = 1, \dots, n, \quad (9)$$

where  $A_i(z, X_1, \dots, X_n) \in \mathbb{C}[z, X_1, \dots, X_n]$  for  $i = 0, \dots, n$  (we suppose that  $A_0$  is a non-zero polynomial).

**Remark 2.3** *It is easy to see that system ((3),(4),(5)) is of the type (9).*

One associates to the system (9) the differential operator

$$D_A = A_0(z, X_1, \dots, X_n) \frac{\partial}{\partial z} + \sum_{i=1}^n A_i(z, X_1, \dots, X_n) \frac{\partial}{\partial X_i}. \quad (10)$$

In our case (i.e. the case of the system (9)) this formula gives exactly the differential operator  $D$  as defined in (7).

**Theorem 2.4 (Nesterenko)** *Suppose that functions*

$$\underline{f} = (f_1(z), \dots, f_n(z)) \in \mathbb{C}[[z]]^n$$

*are analytic at the point  $z = 0$  and form a solution of the system (9). If there exists a constant  $K_0$  such that every  $D$ -stable prime ideal  $\mathcal{P} \subset \mathbb{C}[X'_1, X_1, \dots, X_n]$ ,  $\mathcal{P} \neq (0)$ , satisfies*

$$\min_{P \in \mathcal{P}} \text{ord}_{z=0} P(z, \underline{f}) \leq K_0, \quad (11)$$

*then there exists a constant  $K_1 > 0$  such that for any polynomial  $P \in \mathbb{C}[X'_1, X_1, \dots, X_n]$ ,  $P \neq 0$ , the following inequality holds*

$$\text{ord}_{z=0}(P(z, \underline{f})) \leq K_1(\deg_{\underline{X}'} P + 1)(\deg_{\underline{X}} P + 1)^n. \quad (12)$$

To apply Theorem 2.4 it is sufficient to prove Proposition 2.5 here below.

**Proposition 2.5** *If  $\mathcal{P}$  is a prime ideal of*

$$R = \mathbb{C}[z, X_1, X_2, X_3, Y_{0,1}, \dots, Y_{m-1,m}]$$

*with  $D\mathcal{P} \subset \mathcal{P}$ , then either  $z \in \mathcal{P}$  or  $\Delta = X_2^3 - X_3^2 \in \mathcal{P}$ .*

*Proof of Theorem 2.1 modulo Proposition 2.5.* If we have the result announced in Proposition 2.5, then any prime  $D$ -stable ideal  $\mathcal{P}$  contains the polynomial

$$\Theta := z\Delta = z(X_2^3 - X_3^2). \quad (13)$$

In this case we have obviously

$$\begin{aligned} & \min_{P \in \mathcal{P}} \text{ord}_{z=0} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) \\ & \leq \text{ord}_{z=0} \Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)), \end{aligned}$$

The quantity  $K_0 := \text{ord}_{z=0} \Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z))$  is an absolute constant, in particular independent of  $\mathcal{P}$  (because  $\Theta$  is just a concrete polynomial). Also, the quantity  $K_0$  is finite, because all the functions  $z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)$  are algebraically independent over  $\mathbb{C}$  and for this reason no polynomial vanishes on this set (i.e. in particular,  $\Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z))$  is a non-zero function, analytic at  $z = 0$ ).  $\square$

To prove Proposition 2.5, we describe at first principal  $D$ -stable ideals of  $R$ .

**Lemma 2.6** *There exists only two  $D$ -invariant principal prime ideals of  $R$ , namely, the ideals generated by  $z$  and  $\Delta$ .*

*Proof.* Suppose that  $A \in R$  is any irreducible polynomial with the property that  $A|DA$ . Thus

$$DA = AB, \quad B \in R. \quad (14)$$

We readily verify with the definition of  $D$  that  $\deg_Y DA \leq \deg_Y A$  and  $\deg_z DA \leq \deg_z A$ , hence (14) implies  $B \in \mathbb{C}[X_1, X_2, X_3]$ .

For any  $F \in R$  we define the *weight* of  $F$  as

$$\phi(F) := \deg_t F(z, tX_1, t^2X_2, t^3X_3, t^{2m+2}\underline{Y}).$$

Then  $\phi$  satisfies the following properties:

1. For any  $F \in R$

$$\phi(DF) \leq \phi(F) + 1.$$

2. For any  $F, G \in R$

$$\phi(FG) = \phi(F) + \phi(G).$$

These properties together with (14) imply

$$\phi(A) + \phi(B) = \phi(DA) \leq \phi(A) + 1,$$

hence  $\phi(B) \leq 1$ . Thus  $B \in \mathbb{C}[X_1]$ ,  $\deg B \leq 1$ , i.e.  $B = aX_1 + b$ ,  $a, b \in \mathbb{C}[z]$  and

$$DA = (aX_1 + b)A. \quad (15)$$

Also  $\deg_z A + \deg_z B = \deg_z DA \leq \deg_z A$ , hence  $a, b \in \mathbb{C}$ .

Now we consider another weight  $\phi_2 : R \rightarrow \mathbb{Z}$ . For any  $F \in R$ , we denote

$$\phi_2(F) := \deg_t F(z, tX_1, t^2X_2, t^3X_3, t^{-4}Y_{0,1}, \dots, t^{-4m}Y_{0,m}, t^{-4m+4}Y_{1,m}, \dots, t^{-4}Y_{m-1,m})$$

(i.e. we assign to the variable  $Y_{u,v}$  the weight  $\phi_2(Y_{u,v}) := 4(u - v)$ ). Let  $C$  be the sum of monomials of  $A$  with minimal weight  $\phi_2$ . If we compare the sum of the monomials of weight  $\phi_2(C)$  on both sides of (15) and use the definition of  $D$  we obtain

$$z \frac{d}{dz} C = bC \quad (16)$$

(indeed, for any monomial  $M$  and any differential operator  $D_v$ ,  $v = 1, 3, 5, \dots, m$ , all the non-zero monomials of  $D_v(M)$  have weight  $\phi_2$  strictly bigger than  $\phi_2(M)$ , also the only term in  $D_0$  that does not increase  $\phi_2$  is  $z \frac{d}{dz}$ , hence (16)). Comparing the coefficients on the both sides of (16) we conclude  $b = \deg_z C$ , in particular  $b \in \mathbb{Z}$ .

Substituting  $X_1 = E_2(z)$ ,  $X_2 = E_4(z)$ ,  $X_3 = E_6(z)$ ,  $Y_{u,v} = g_{u,v}(z)$ ,  $0 \leq u < v \leq m$  in (15) we obtain

$$(aE_2(z) + b) A(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) \\ = DA(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)). \quad (17)$$

Let

$$A(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) = cz^M + (\text{terms of order } > M),$$

$c \neq 0$ , be the (first term of the) Taylor series of  $A(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z))$ . In view of the property 8 we have

$$DA(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) = cMz^M + (\text{terms of order } > M).$$

Using the Taylor series for  $E_2$ , (1), notably the fact that  $E_2(z) = 1 +$  terms of order  $> 1$ , we readily deduce from (17)

$$(a + b)cz^M + (\text{terms of order } > M) = cMz^M + (\text{terms of order } > M).$$

Comparing coefficients with  $z^M$  in the l.h.s. and in the r.h.s. of (17) and simplifying out  $c$  we readily deduce  $a + b = M$ . We have already established  $b \in \mathbb{N}$ . Obviously,  $M \in \mathbb{N}$  (as it is a degree in a Taylor series). We conclude  $a \in \mathbb{Z}$ .

So we have established that coefficients  $a, b$  involved in (15) are in fact integers.

Note that

$$D(\Delta^{-a}z^{-b}) = (-aX_1 - b)\Delta^{-a}z^{-b}. \quad (18)$$

We denote

$$S(z, E_2, E_4, E_6, g_{0,v}, \dots, g_{v-1,v}) := A(z, E_2, E_4, E_6, g_{0,v}, \dots, g_{v-1,v}) \Delta^{-a}z^{-b}. \quad (19)$$

Applying the differential operator  $D$  to the r.h.s. of (19) and using (15), (18) we find out

$$DS = 0.$$

Using (8) on the latter equality we conclude

$$\frac{d}{dz} S(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) = 0,$$

hence

$$S(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) \in \mathbb{C}.$$

Recall that functions  $z, E_2, E_4, E_6, g_{0,v}, \dots, g_{v-1,v}$  are all algebraically independent over  $\mathbb{C}$ , see [2] page 2. For this reason we deduce  $S[X_0, X_1, X_2, X_3, \underline{Y}] \in \mathbb{C}$  and thereby

$$A = \Delta^a z^b.$$

If we suppose that  $A$  is irreducible, we obtain immediately that either  $(a, b) = (1, 0)$  or  $(a, b) = (0, 1)$ .  $\square$

*Proof of Proposition 2.5.* We consider the following nested sequence of rings

$$\begin{aligned} \mathbb{C}[z, \underline{X}] &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}] \subset \mathbb{C}[z, \underline{X}, Y_{0,1}, Y_{2,3}] \subset \mathbb{C}[z, \underline{X}, Y_{0,1}, Y_{1,3}, Y_{2,3}] \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, Y_{0,3}, Y_{1,3}, Y_{2,3}] \subset \cdots \subset \mathbb{C}[z, \underline{X}, Y_{0,1}, \dots, Y_{m-3, m-2}] \\ &\quad \subset \mathbb{C}[z, \underline{X}, Y_{0,1}, \dots, Y_{m-3, m-2}, Y_{m-1, m}] \\ &\quad \subset \mathbb{C}[z, \underline{X}, Y_{0,1}, \dots, Y_{m-3, m-2}, Y_{m-2, m}, Y_{m-1, m}] \\ &\quad \subset \cdots \subset \mathbb{C}[z, \underline{X}, Y_{0, m}, \dots, Y_{m-1, m}] = R. \end{aligned} \quad (20)$$

We readily verify with the definition of  $D$  that every term  $R_i$  appearing in the chain (20) satisfies  $DR_i \subset R_i$ .

Let  $\mathcal{P} \subset R$  be a prime ideal of  $R$  satisfying  $D\mathcal{P} \subset \mathcal{P}$ . If  $\mathcal{P} \cap \mathbb{C}[z, \underline{X}] \neq \{0\}$ , it contains a polynomial  $z\Delta$  as shown in [4][Theorem 1.4]. So everything is proved in this case. We suppose henceforth  $\mathcal{P} \cap \mathbb{C}[z, \underline{X}] = \{0\}$ .

We proceed with recurrence. As we suppose  $\mathcal{P} \neq \{0\}$  and  $\mathcal{P} \cap \mathbb{C}[z, \underline{X}] = \{0\}$ , we find in the chain (20) at some step an extension of rings  $R_i \subset R_{i+1}$  satisfying  $\mathcal{P} \cap R_i = \{0\}$  and  $\mathcal{P} \cap R_{i+1} \neq \{0\}$ . In this case the ideal (of the ring  $R_{i+1}$ )  $\mathcal{P} \cap R_{i+1} \neq \{0\}$  is a principal one, because we add exactly one variable at each step in the chain (20), i.e.  $\text{tr.deg.}_{R_i} R_{i+1} = 1$ . Hence  $\mathcal{P} \cap R_{i+1}$  is a  $D$ -stable principal ideal (of the ring  $R_{i+1}$ , and also this ideal generates a principal  $D$ -stable ideal of the ring  $R$ , because  $D$ -stability of a principal ideal means exactly the condition  $Q|DQ$  on a generator of the ideal). We deduce with Lemma 2.6 that  $z\Delta \in \mathcal{P} \cap R_{i+1} \subset \mathcal{P}$ , Q.E.D.  $\square$

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